The simplest Regge calculus model in the canonical form

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Abstract

Dynamics of a Regge three-dimensional (3D) manifold in a continuous time is considered. The manifold is closed consisting of the two tetrahedrons with identified corresponding vertices. The action of the model is that obtained via limiting procedure from the general relativity (GR) action for the completely discrete 4D Regge calculus. It closely resembles the continuous general relativity action in the Hilbert-Palatini (HP) form but possesses finite number of the degrees of freedom. The canonical structure of the theory is described. Central point is appearance of the new relations with time derivatives not following from the Lagrangian but serving to ensure completely discrete 4D Regge calculus origin of the system. In particular, taking these into account turns out to be necessary to obtain the true number of the degrees of freedom being the number of linklengths of the 3D Regge manifold at a given moment of time.

Regge calculus recently remains probably the most promising tool in the attempts to quantise gravity [1, 2, 3]. It's usual application is that in the form of simplicial minisuperspace [4] used in finding a wavefunction of the universe by summation over histories in the Feynman path integral [5]. The advantage of the minisuperspace is that it possesses a finite number of the degrees of freedom. This makes path integral well-defined as usual multiple integral.

In the simplicial minisuperspace framework the area variables on the triangles seem to be more natural than the linklengths [6]; at the same time neighbouring area bivectors are not independent. One can view the long-distance effects such as gravity waves as a result of correlations between the neighbouring areas for if these were independent, the theory would be locally trivial by Einstein eqs. in the Regge calculus framework. It was shown in [7] how Regge calculus reproduces gravity waves of GR in the long wavelength limit for the periodic Regge manifold of a specific form.

Now introduce the model consisting of the two evolving in the continuous time tetrahedrons with common face ABC and identified fourth vertices D and D'. This is an abstract situation when the long distance effects (gravity waves) are absent simply because the "long distances" themselves are absent. Thereby we concentrate on the pure local effects steming from the neighbourhood of the only few triangles.

In the given paper we formulate this model in the canonical form. It turns out that using Regge calculus in the area bivector – connection variables [8] leads to the form of the action which notationally has almost complete correspondence with the HP form of the continuum GR action. The advantage of the connection variables is that these are independent ones, at the same time their being excluded from the action with the help of eqs. of motion gives exactly Regge action in terms of linklengths. The considered action is the limiting form of the full 4D discrete Regge calculus action obtained by passing to the continuous time. The continuous time formalism provides the strict framework for the canonical formalism [9] and subsequent canonical (Dirac) quantisation.

Central point of the paper is appearance of the additional constraints not present in the original Lagrangian and simultaneous extension of the phase space necessary to obtain the true number of the degrees of freedom. These constraints are continuous time limit form of the relations showing that definite bivectors correspond to triangles which form 4D Regge manifold. It was shown in [9] that the constraints of such type follow from the Lagrangian in the case when the number of the triangles N_2 is not smaller than the number of links N_1 in the 3D leaf and do not require to be set additionally in such case. However, in our case $N_2 = 4$, $N_1 = 6$, and the situation is different.

Let us first briefly repeat the formulas of [9] of interest. If the triangle is spanned by two links with vectors l_1^a, l_2^a (in the local frame of one of the tetrahedrons sharing this triangle) let us define the bivector of the triangle via

$$\pi_{ab} = \pm\varepsilon_{abcd} l_1^c l_2^d \tag{1}$$

where a, b, c,... $= 0, 1, 2, 3$ are local indices, ε_{abcd} is completely antisymmetric tensor, $\varepsilon_{0123} = +1$. Let us denote the vertices of the both tetrahedrons by μ, ν, λ,... $= 0, 1, 2, 3$ and, by the same symbols, the opposite triangles. A sign in the definition of the

bivector π^μ of the triangle μ is chosen so that $\pi^0 + \pi^1 + \pi^2 + \pi^3 = 0$. It is convenient to write $v_\mu n_{\mu\nu} dt$ for the infinitesimal bivector of the timelike triangle formed by the link $(\mu\nu)$ which connects the vertices μ and ν and by the infinitesimal timelike link $(\mu\mu')$ connecting μ with it's image μ' in the next time leaf (taken at the moment $t + dt$ of the world time). The v_μ defines the scale of (local) time at the point μ. In [9] the π and n bivectors were shown in general case to be subject to a system of bilinear constraints guaranteing their being bivectors of some Regge manifold. In our simple case we can write out explicit formula for $n_{\mu\nu}$ in terms of π^μ and choose the time scale parameter v_μ so that

$$n_{\mu\nu} = \sum_{\lambda,\rho} \varepsilon_{\mu\nu\lambda\rho} \left(u^\lambda_\mu \pi^\rho + \frac{1}{2}[\pi^\lambda, \pi^\rho] \right) \tag{2}$$

where u^λ_μ are parameters (independent variables). The timelike triangle $(\mu\mu'\nu)$ is shared (in the 4D manifold) by the two tetrahedrons $(\mu\mu'\nu\lambda)$ and $(\mu\mu'\nu\rho)$ where λ, ρ are the two rest vertices other than μ, ν. The two connection SO(3,1) matrices (SO(4) in the Euclidean case) live on these tetrahedrons and define the curvature matrix on the triangle $(\mu\mu'\nu)$ — $R_{\mu\nu}$. The connection on $(\mu\mu'\nu\lambda)$ will be denoted as $\Omega_{\mu\rho}$ with ordered pair of indices $\mu\rho$ denoting the tetrahedron spanned by the link $(\mu\mu')$ and triangle $\rho = (\mu\nu\lambda)$. Then

$$R_{\mu\nu} = \frac{1}{2} \sum_{\lambda,\rho \neq \mu,\nu} \left(\bar{\Omega}_{\mu\rho} \Omega_{\mu\lambda} \right)^{\varepsilon_{\mu\nu\lambda\rho}} \tag{3}$$

where overlining means Hermitean conjugation (in fact, this reduces to only one term, e.g. $R_{01} = \bar{\Omega}_{03}\Omega_{02}$). The $\Omega_{\mu\nu}$ were shown [9] to be parameterised as follows:

$$\Omega_{\mu\nu} = \Omega_\nu \exp\left(\varphi^\nu_\mu \pi^\nu + {}^*\!\varphi^\nu_\mu {}^*\!\pi^\nu\right) \tag{4}$$

where

$${}^*\!\pi_{ab} = \frac{1}{2} \varepsilon_{abcd} \pi^{cd} \tag{5}$$

and φ^ν_μ, ${}^*\!\varphi^\nu_\mu$ are parameters (independent variables). The form (4) is necessary to ensure finiteness of the action: the deficit angle which residues on spacelike triangle is generally not infinitesimal, and (4) provides up to $O(dt)$ cancellation of the contributions into action from these triangles located inside the infinitesimal prism with bases $(\mu\nu\lambda)$ and $(\mu'\nu'\lambda')$; on the other hand, namely the fact that exponential in (4) cannot be absorbed into Ω_ν for different μ, i.e. $\Omega_{\mu_1\nu} \neq \Omega_{\mu_2\nu}$ for $\mu_1 \neq \mu_2$, leads to independence between $R_{\mu\nu}$ and $R_{\nu\mu}$: $R_{\mu\nu} \neq R^{\pm 1}_{\nu\mu}$, so that the planes of the triangles $(\mu\mu'\nu)$ and $(\mu\nu'\nu)$ around which $R_{\mu\nu}$ and $R_{\nu\mu}$ rotate do not coincide and therefore the links $(\mu\mu')$ and $(\nu\nu')$ (analogs of the lapse-shift vectors at the different vertices) are independent of each other as these should be. It is the remaining factor Ω_ν which can be attributed to the triangle ν when we speak of the geometry induced in the 3D leaf and we can speak of it as that rotating from one tetrahedron to another one in this leaf. Defining scalar product of bivectors A^{ab} and B^{ab} as

$$A \circ B = \frac{1}{2} \mathrm{tr} A\bar{B} \tag{6}$$

we can write out the Lagrangian:

$$L = \sum_\mu \pi^\mu \circ \bar{\Omega}_\mu \dot{\Omega}_\mu - \sum_\mu v_\mu H_\mu - h \circ \sum_\mu \pi^\mu - \tilde{h} \circ \sum_\mu \Omega_\mu \pi^\mu \bar{\Omega}_\mu \tag{7}$$

(dot means time derivative) where

$$\begin{aligned} H_\mu &= \sum_\nu |n_{\mu\nu}| \arcsin \frac{n_{\mu\nu} \circ R_{\mu\nu}}{|n_{\mu\nu}|} + \varphi^\nu_\mu \pi^\nu \circ \Delta_\mu \pi^\nu & (8) \\ \Delta_\mu \pi^\nu &\stackrel{\text{def}}{=} \sum_{\lambda,\rho} \varepsilon_{\mu\nu\lambda\rho} n_{\mu\lambda} & (9) \end{aligned}$$

The $\Delta_\mu \pi^\nu$ defines variation $v_\mu \Delta_\mu \pi^\nu dt$ of the bivector π^ν due to the shifting the vertex μ to μ'. By varying L in the antisymmetric h and $\tilde{h}$ we get the Gauss law in each tetrahedron expressing closeness of the tetrahedron surface.

Now consider the constraints not following from the Lagrangian. Among these there are the following ones:

$$\pi^\mu * \pi^\nu \stackrel{\text{def}}{=} \pi^\mu \circ ({}^*\pi^\nu) = 0 \tag{10}$$

ensuring π^μ being face bivectors of some tetrahedron. The four such constraints with $\mu = \nu$ are, in fact, I class ones: these appear added to the Lagrangian at the following symmetry transformation:

$$\begin{aligned} \Omega_\mu &\to \Omega_\mu \exp(\zeta^\mu \, {}^*\pi^\mu), & (11) \\ {}^*\varphi^\mu_\nu &\to {}^*\varphi^\mu_\nu - \zeta^\mu, & (12) \end{aligned}$$

ζ^μ being parameters. Note that the other I class constraints, the Gauss law, serve to maintain invariance at the usual local frame rotations:

$$\begin{aligned} \pi^\mu &\to U \pi^\mu \bar{U}, & (13) \\ \Omega_\mu &\to O \Omega_\mu \bar{U}, & (14) \\ h &\to h - \bar{U}\dot{U}, & (15) \\ \tilde{h} &\to \tilde{h} + \bar{O}\dot{O}, & (16) \end{aligned}$$

$O, U \in$ SO(3,1) (SO(4) in the Euclidean case). These constraints being I class can be displayed also by their commutativity with other constraints (including the Hamiltonian) w.r.t. the Poisson brackets (PB) defined for any function f of π, Ω so that

$$\dot{f} = \{H, f\}. \tag{17}$$

This leads to the following definition:

$$\{f, g\} = \pi \circ [f_\pi, g_\pi] + f_\pi \circ \bar{\Omega} g_\Omega - g_\pi \circ \bar{\Omega} f_\Omega. \tag{18}$$

Here indices π, Ω mean corresponding derivative; the derivatives over Ω times $\bar{\Omega}$ and over π are assumed to be symmetrised; the summation over the pairs (π^μ, Ω_μ) is implied.

The two independent constraints among (10) with $\mu \neq \nu$ are not I class. As those we can choose, e.g., $\pi^1 * \pi^2$ and $\pi^2 * \pi^3$ (such as $\pi^0 * \pi^1$ and $\pi^2 * \pi^3$ are not independent modulo I class costraints).

Thus far we have 16 first class constraints (including the 12 Gauss law components) and 6 second class constraints (including the four H_ν). Without taking the constraints into account the phase space of π, Ω would correspond to 24 degrees of freedom; taking the above constraints into account diminishes this number to 24 - 16 - $\frac{1}{2}6 = 5$. It does not coincide with the number of leaf linklengths which is 6. (Note that the nondynamical variables u^ν_μ, φ^ν_μ, ${}^*\varphi^\nu_\mu$ enter L nonlinearly and are given as implicit functions of π, Ω by eqs. of motion for them,

$$\frac{\partial \mathcal{H}_\mu}{\partial (u^\nu_\mu, \varphi^\nu_\mu, {}^*\varphi^\nu_\mu)} = 0 \tag{19}$$

).

This disagreement takes place because we have not completely specified our continuous time system as limiting case of the completely discrete 4D Regge manifold. Indeed, whereas the constraints (10) define the leaf tetrahedron at the time t, and expression (2) for $n_{\mu\nu}$ guarantees that the $n_{\mu\nu}$ are the bivectors of rigid 4D structure filling in the spacetime between the two successive leaves at t and $t + dt$, there is also need in the relations ensuring possibility of glueing together the two successive such structures: one between t and $t + dt$ leaves and another one, say, between $t + dt$ and $t + 2dt$ leaves. As such relations, one can set continuity of scalar π^μ bilinears on the junction leaf at $t + dt$: on the one hand, to form the boundary of the 4D submanifold between t and $t + dt$, such bivectors should be

$$\pi^\mu(t) + \sum_\nu v_\nu \Delta_\nu \pi^\mu dt;$$

on the other hand, in the 4D submanifold between $t + dt$ and $t + 2dt$ this is simply $\pi^\mu(t + dt)$. Thus,

$$\frac{d}{dt}(\pi^\mu \circ \pi^\nu) = \Delta(\pi^\mu \circ \pi^\nu)(\Delta \stackrel{\text{def}}{=} \sum_\nu v_\nu \Delta_\nu). \tag{20}$$

At $\mu = \nu$ such relation follows from the eqs. of motion, namely, gets added to the Lagrangian upon the following variation of variables:

$$\Omega_\mu \quad \rightarrow \quad \Omega_\mu \exp(\xi^\mu \pi^\mu), \tag{21}$$

$$\varphi^\mu_\nu \quad \rightarrow \quad \varphi^\mu_\nu - \xi^\mu. \tag{22}$$

As the rest independent (modulo Gauss law) relations may be chosen, e.g., those for $d(\pi^1 \circ \pi^2)/dt$ and $d(\pi^2 \circ \pi^3)/dt$.

In principle, one can exclude the derivatives $\dot{\pi}^\mu$ with the help of eqs. of motion thus converting these relations into constraints on π, Ω in the usual sense. But this introduces a nonequivalence of time and space additional to the already existing nonequivalence connected with continuity of time and discreteness of space. Aiming at formulating the theory maximally symmetrically we choose to add these relations to the Lagrangian with the help of Lagrange multipliers, ψ_{12} and ψ_{23}, just as we (implicitly) do so for

other bilinear constraints on bivectors not containing the time derivatives. The new Lagrangian up to the full derivative reads:

$$\mathcal{L} = L + \dot{\psi}_{12}\pi^1 \circ \pi^2 + \dot{\psi}_{23}\pi^2 \circ \pi^3 + \psi_{12}\Delta(\pi^1 \circ \pi^2) + \psi_{23}\Delta(\pi^2 \circ \pi^3). \tag{23}$$

Thus we get the new dynamical variables. In the Hamiltonian formalism the phase space is extended by the two new canonical pairs ($\tilde{\psi}^{12}$, ψ_{12}) and ($\tilde{\psi}^{23}$, ψ_{23}), $\tilde{\psi}^{\mu\nu}$ being conjugate momenta subject to the two new constraints:

$$\tilde{\psi}^{12} - \pi^1 \circ \pi^2 = 0, \quad \tilde{\psi}^{23} - \pi^2 \circ \pi^3 = 0. \tag{24}$$

Besides, Hamiltonian constraints are modified:

$$H_\mu \to \mathcal{H}_\mu = H_\mu - \psi_{12}\Delta_\mu(\pi^1 \circ \pi^2) - \psi_{23}\Delta_\mu(\pi^2 \circ \pi^3). \tag{25}$$

The PB for the case of ψ, $\tilde{\psi}$ dependence are modified in obvious way. The $\mathcal{H}_\mu$, $\pi^1 * \pi^2$, $\pi^2 * \pi^3$ and new constraints (24) generally form the second class system since determinant of their PB is not identical zero. The overall effect of extending phase space is therefore enhancing the number of the degrees of freedom by 1, so that this number is just 6, the number of leaf linklengths.

The two variables $\psi_{\mu\nu}$ can be excluded solving two of the four Hamiltonian constraints $\mathcal{H}_\mu$. This leaves us with only two Hamiltonian constraints on the phase space of purely (π, Ω) pairs.

The system obtained strongly remind the usual HP form of the continuum GR (see, e.g., review [10]). If (purely formally, forgetting for a moment that the system is strongly nonperturbative) we expand the action over small ω_μ, φ^μ_ν, ${}^*\varphi^\mu_\nu$ (where $\Omega_\mu = \exp \omega_\mu$), we reproduce the kinetic term $\pi^\mu \circ \dot{\omega}_\mu$; analog of the vector (coefficients at u^μ_ν), Hamiltonian (coefficients at v_μ) and Gauss law constraint (coefficients at h, $\tilde{h}$); the constraint ensuring the tetrad form of π^μ ($\pi^\mu * \pi^\nu = 0$). The difference is that, on the one hand, differentiating the constraint $\pi^\mu * \pi^\nu$ now does not yield the new constraint (rather defines some Lagrange multipliers) and, on the other hand, the new variables $\psi_{\mu\nu}$, $\tilde{\psi}^{\mu\nu}$ and related new constraints arise. Besides that, analogs of the vector and Hamiltonian constraints are not first class (do not commute mutually and with other constraints).

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